A Continuous-time Theory of Currency Substitution

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Abstract

We analyze the coexistence of cash (fiat money) and privately-issued currencies (crypto-currencies) in a dynamic model where all factors of production are paid in fiat money. This introduces a cash-in-advance constraint that affects both consumption and investment, leading to non-neutrality of money. Crypto-currencies add distortions through labor reallocation and transaction fees. Using flexible utility specifications, we explore the impact of substitutability between money and crypto-purchased goods. Our main result is that an increase in the money supply raises inflation and shifts labor allocation, affecting growth dynamics. While broader economic variables remain stable, real wages are highly sensitive to changes in consumer preferences and crypto-fees, underscoring the impact of private digital currencies on the economy's long-term trajectory.

Keywords : Fiat money, crypto-currency and non-neutrality, cash-in-advance, growth theory.

1 Introduction

We study the coexistence of fiat money and crypto-currencies in a dynamic model where all production factors are legally required to be compensated in fiat money, the legal tender. This legal requirement for all transactions to be conducted in a government-backed currency is equivalent to a comprehensive cash-in-advance (CIA) constraint, which mandates that money must cover both consumption expenditures and investment in physical capital. The introduction of physical capital investment and dynamic considerations allows us to capture the non-neutrality of both fiat money and crypto-currencies, which does not occur in models where both media of exchange serve purely transactional functions.

The CIA constraint also extends to exchanges of fiat money for crypto-currencies, which are then used to purchase crypto-paid consumption goods. We model the representative consumer's preference for either form of payment through different mathematical specifications of the instantaneous utility function. Initially, the consumer derives utility by combining money-purchased

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and crypto-purchased goods through a Cobb-Douglas function. Later, we generalize our analysis by considering a Constant Elasticity of Substitution (CES) framework, which offers more flexibility to study the limiting cases of complementarity and substitutability between fiat money and crypto-currencies.

Additionally, we abstract from explicitly modeling the pecuniary and non-pecuniary benefits of crypto-purchased goods, as was done in our static framework. Instead, we assume that consumers perceive money-purchased and crypto-purchased goods as yielding different utilities, even though firms view these goods as identical from a production standpoint.

Our first key result is that money is neither neutral nor super-neutral in our system. We provide both analytical and numerical derivations of this non-neutrality in section 8, within both the neoclassical and endogenous growth frameworks. A shock in the money supply growth rate propagates through the economy by raising inflation, which in turn reduces real consumption growth due to the increased cost of holding money. This affects capital accumulation and labor allocation as the CIA constraint forces adjustments in consumption and investment. The impact of this shock is amplified by the degree of substitutability between money-purchased and crypto-purchased goods, leading to shifts in real wages and sectoral labor allocation.

The introduction of crypto-currencies alters the dynamics of monetary non-neutrality in our model. While money is non-neutral due to the cash-in-advance (CIA) constraint on capital purchases, where inflation raises the cost of holding money, crypto-currencies add another layer of distortion. Labor is diverted from goods production to the operation of crypto-exchange platforms, reducing the capital-labor ratio and leading to under-accumulation of capital and lower consumption levels. Furthermore, crypto-currency fees distort the balance between money-purchased and crypto-purchased goods, creating additional inefficiencies in the steady state economy.

In subsection 8.3, we illustrate how key economic variables respond to changes in the elasticity of substitution between money-purchased and crypto-purchased goods over time. Despite steady growth in capital, output, and consumption, the elasticity of substitution has little effect on these broader growth trends, indicating that technological progress is the main driver of long-term economic growth. However, real wages are much more sensitive, rising faster when crypto-purchased goods are more easily substituted for money-purchased goods, as labor is allocated more efficiently. In contrast, with stronger complementarity between goods, wage growth is slower. Price levels remain stable, reflecting the relatively small impact of substitution on overall cost structures. These results highlight that while broader growth is driven by technology, labor markets and wages are more responsive to changes in consumer preferences.

This framework allows us to analyze how fiat money and crypto-currencies interact within an economy and how legal and economic constraints influence the dynamic allocation of resources across different sectors. Our analysis provides valuable insights into how monetary policy and exchange platform fees impact the long-run growth trajectory of an economy, offering new perspectives on the role of private digital currencies in modern monetary systems.

The remainder of the paper progresses as follows. We first present the connection between our results and the existing literature in section 2. We then present our dynamic model in section 3. Then, we explore the solutions emerging from the optimization problem in section 4. Finally, we provide a numerical analysis to study the response of the endogenous variables following a shock to the exogenous constants in the final section.

2 Related Literature

This paper builds on the literature of private currencies, monetary economics, and currency competition. We start with the monetary model of Marchiori (2021), where a cash-in-advance constraint requires consumers to exchange part of their money holdings for crypto-currencies to purchase specific goods. While Marchiori (2021) focuses on Bitcoin supply growth, our analysis examines how exchange platform fees and consumer preferences for crypto-purchased goods affect the economy. In this sense, we offer a partial equilibrium analysis, abstracting from the mining sector, similar to other models in the literature (see Schilling & Uhlig (2019b), Lotz & Vasselin (2019), Benigno et al. (2022)).

Another important contribution to the literature is highlighting key policy implications regarding the crypto-money linkage. For instance, Schilling & Uhlig (2019a) argue that welfare remains unaffected in a monetary model with cryptocurrency price dynamics. However, we provide evidence in Table 4 and Table 9 that a shock to exchange platform transaction fees can distort welfare. The welfare level varies depending on the functional forms of the utility function. Fernández-Villaverde & Sanches (2019) characterizes the equilibrium welfare level in the presence of a private currency as wasteful, where the authority fails to provide the necessary amount of money for transactions. Our model does not include variables to make such a statement.

3 Model Setup

3.1 Cash-in-advance and dynamic budget constraints

Time is continuous and indexed by $t \in [0, \infty)$. The economy is populated by a constant number of L identical households purchasing consumption goods with a combination of fiat money issued by the government (henceforth, money) and crypto-currency purchased on exchange platforms. Each consumer is infinitely-lived and maximizes intertemporal lifetime utility

$$U \equiv \int_0^\infty e^{-\rho t} \ln \left[u \left(c_t^m, c_t^x \right) \right] dt = \int_0^\infty e^{-\rho t} \ln \left[\left(c_t^m \right)^\theta \left(c_t^x \right)^{1-\theta} \right] dt, \tag{1}$$

where $\rho > 0$ is the utility discount rate, and c_t^m and c_t^x indicate units of the consumption good purchased at time t by means of money and crypto-currency, respectively.

Money is printed costlessly by the government and transferred to households via lump-sum transfers. The single consumer uses money to purchase new capital, to directly purchase c_t^m units of output or to purchase units of crypto-currency that are then used to purchase c_t^x units of output.

Denoting aggregate real investment in physical capital by $\dot{K}_t$, aggregate nominal money by M_t and the aggregate units of purchased crypto-currency by S_t, the CIA constraint that applies to money holdings at the individual level reads

$$\frac{M_t}{L} = P_t \frac{\dot{K}_t}{L} + P_t c_t^m + Q_t \left(1 + \delta_t\right) \frac{S_t}{L}, \tag{2}$$

where Q_t is the nominal exchange rate between money and crypto-currency and δ_t are fees paid by the household to acquire the crypto-currency from exchange platforms – i.e., the units of money needed by households to purchase one unit of crypto on the market are $Q_t(1 + \delta_t)$. The budget for crypto-paid consumption goods is subject to the parallel *crypto-CIA constraint*

$$\frac{S_t}{L} = P_t^* c_t^x. \tag{3}$$

From (2) and (3), the *combined CIA constraint* reads

$$\frac{M_t}{L} = P_t \frac{\dot{K}_t}{L} + P_t c_t^m + Q_t \left(1 + \delta_t\right) P_t^* c_t^x. \tag{4}$$

Each household supplies labor (inelastically) to firms and owns a fraction $1/L$ of the existing capital stock K_t that firms use as an input in goods' production. The household dynamic budget constraint in money terms reads

$$P_t \frac{\dot{K}_t}{L} + \frac{\dot{M}_t}{L} = w_t + r_t \frac{K_t}{L} + \frac{V_t}{L} - P_t c_t^m - Q_t \left(1 + \varphi_t\right) P_t^* c_t^x \tag{5}$$

where w_t is the monetary wage rate, r_t is the rate of return to capital in terms of money, V_t equals aggregate lump-sum transfers from the government to all households. We henceforth normalize total population (workforce) to unity, $L = 1$, and transform (5) in real terms by defining real money as $m_t = M_t/P_t$, real money transfers as $v_t = V_t/P_t$, and the real exchange rate $q_t = \frac{Q_t P_t^*}{P_t}$, from which we obtain

$$\dot{K}_t + \dot{m}_t = \frac{1}{P_t} \left(w_t + r_t K_t\right) + v_t - c_t^m - q_t \left(1 + \delta_t\right) c_t^x - m_t \pi_t. \tag{6}$$

Similarly, the combined CIA constraint (4) can be rewritten as

$$m_t = \dot{K}_t + c_t^m + q_t \left(1 + \delta_t\right) c_t^x. \tag{7}$$

The household problem consists of maximizing present-value utility (1) subject to the constraints (6) and (7). As usual in the literature, we postulate that CIA constraints hold as strict equalities – i.e., both the CIA constraint on money (2) and the CIA constraint on the crypto-currency (3) are binding because both currencies are strictly dominated by physical capital in terms of rate of returns. We are thus focusing on environments where the rate of money deflation ($-\pi_t = -\dot{P}_t/P_t$) is smaller than the market rental rate of capital – which is the case in any economy with positive inflation – and where agents do not accumulate crypto-currency as an asset – which is guaranteed by a similar return-dominance condition that we will formulate and impose ex post via parameter restrictions.

3.2 Production

All consumption goods are produced with the same constant returns to technology by a competitive sector: total final output equals $Y_t = F\left(K_t, a_t L_t^y\right)$, where L^y is labor employed in goods production and a_t is labor productivity. Real output is sold to households either as consumption or as new physical capital:

$$F\left(K_t, a_t L_t^y\right) = c_t^m + c_t^x + \dot{K}_t = C_t + \dot{K}_t \tag{8}$$

where we have defined $C_t \equiv L c_t^m + L c_t^x$ as aggregate real consumption and L is normalized to unity. From the producers point of view, there is perfect substitutability among the three uses of the final good, which implies price equalization: by no-arbitrage logic, each unit of output must yield P_t units of money. New capital and money-paid consumption goods indeed have the same money price P_t. For crypto-paid consumption goods, the production sector will charge a crypto-price P_t^* that generates the same unit revenue after conversion.

After selling c_t^x units against crypto-currency, producers will convert the associated crypto-payments into money so as to compensate production factors. Converting the $P_t^* c_t^x$ units of crypto received from customers into money involves paying a proportional fee to exchange platforms. We set the fee rate for firms equal to δ_t, the same fee rate paid by households acquiring the crypto-currency. The net money revenue from selling crypto-paid goods thus equals $Q_t\left(1-\delta_t\right) P_t^* c_t^x$ in terms of money. By no-arbitrage with the revenue that firms would obtain by selling the same units against money, $P_t c_t^x$, it follows that the crypto-price of crypto-purchased goods equals

$$P_t^* = \frac{P_t}{Q_t\left(1-\delta_t\right)}. \tag{9}$$

The total profits of the final sector in terms of money can thus be written as $P_t F\left(K_t, a_t L_t^y\right) - r_t K_t - w_t L_t^y$, and constant returns to scale imply the zero-profit condition

$$P_t F\left(K_t, a_t L_t^y\right) = r_t K_t + w_t L_t^y = P_t C_t + P_t \dot{K}_t. \tag{10}$$

Note that the no-arbitrage condition (9) implies that the real exchange rate equals

$$q_t = \frac{Q_t P_t^*}{P_t} = \frac{1}{1-\delta_t}, \tag{11}$$

so that positive growth in exchange fees implies a real appreciation of the crypto-currency.

3.3 Exchange platform

We model the exchange platform as a competitive sector where an indefinite number of 'crypto-exchange firms' provide services to consumers and firms and bear the cost of validating these currency transactions. In this model, validation is the activity that crypto-exchange firms must perform in every exchange operation between crypto-currency and money – which includes both selling the crypto-currency to consumers and repurchasing it from final producers. The exchange platform as a whole trades $S_t = P_t^* c_t^x$ units of the crypto-currency and employs $1 - L_t^y$ workers to perform validation activities. Since labor is homogeneous and fully mobile between the final goods' production sector and the exchange platform, the wage rate w_t will be equalized between these sectors.

At the aggregate level, the *monetary inflows* of the exchange platform are represented by fees charged on consumers selling money against crypto – that is, money inflows for the platform – and by fees charged on producing firms that sell crypto against money – that is, money retained from the outflows reaching final goods' producers:

$$\begin{aligned} \text{Platform money inflows} &= QP_t^* \left(1+\delta_t\right) c_t^x - QP_t^* \left(1+\delta_t\right) c_t^x = \\ &= 2\delta_t \cdot Q_t \cdot \underbrace{P_t^* c_t^x}_{S_t}. \end{aligned} \tag{12}$$

The monetary outflows of the exchange platform equal the wage bill, $w_t \left(1 - L_t^y\right)$. Zero profits in the crypto-sector thus require

$$2\delta_t Q_t \cdot \underbrace{P_t^* c_t^x}_{S_t} = w_t \left(1 - L_t^y\right). \tag{13}$$

There are many ways to model the behavior of crypto-exchange firms consistently with the above zero-profit condition. We leave this part of the model unspecified for the sake of generality: as we show in sections 5 and 6, many relevant results can be established, for different variants of the model, by simply imposing the zero-profit condition (13) without assuming a specific technology for the exchange platform. Further results for the main variants of the model – the 'neoclassical case' and the 'AK case' – will be obtained later under specific technology assumptions for both goods production and exchange platform.

3.4 Aggregate constraints and equivalence

This subsection briefly (i) derives the aggregate constraint of the economy in money terms and (ii) verifies the equivalence between the CIA constraint imposed on expenditures (7) and the legal requirement that all factor incomes must be paid using the legal tender.

(i) Aggregate constraint of the economy in money terms. By combining the zero profit condition of the production sector (10) with the household budget constraint (6), we obtain

$$\dot{K}_t = F\left(K_t, a_t L_t^y\right) + \frac{w_t \left(1 - L_t^y\right)}{P_t} - c_t^m - q_t \left(1+\delta_t\right) c_t^x + \left(v_t - m_t \pi_t - \dot{m}_t\right) \tag{14}$$

Using the definition of real exchange rate $q_t = \frac{Q_t P_t^*}{P_t}$, we can rewrite the zero profit condition for the exchange platform (13) as

$$2\delta_t q_t P_t c_t^x = w_t \left(1 - L_t^y\right). \tag{15}$$

Substituting (15) in (14) and rearranging terms yields

$$\dot{K}_t = F\left(K_t, a_t L_t^y\right) - c_t^m - q_t \left(1-\delta_t\right) c_t^x + \left(v_t - m_t \pi_t - \dot{m}_t\right)$$

which, after substituting $q_t(1-\delta_t)=1$ from (11), becomes

$$P_t\dot{K}_t + P_t c_t^m + P_t c_t^x = P_t F\left(K_t, a_t L_t^y\right) + P_t\left[v_t - m_t\pi_t - \dot{m}_t\right]. \tag{16}$$

Expression (16) is the aggregate resource constraint of the economy in money terms. In real terms, it reduces to the goods market clearing condition because, under a binding CIA constraint, real money transfers v_t represent the increase in real money holdings,

$$v_t = \frac{\dot{M}_t}{P_t} = \frac{\dot{M}_t}{M_t}\cdot m_t = \left(\frac{\dot{m}_t}{m_t}+\pi_t\right)\cdot m_t = \dot{m}_t + m_t\pi_t, \tag{17}$$

so that the last term in square brackets in (16) cancels out, and dividing boths sides by P_t yields the market clearing condition (8).

(ii) Equivalence between CIA constraint and legal requirement on factor payments. Recalling the CIA constraint in real terms, use (11) to rewrite (7) as

$$m_t = \dot{K}_t + c_t^m + \frac{1+\delta_t}{1-\delta_t}c_t^x, \tag{18}$$

which, after some manipulation, yields

$$m_t = \dot{K}_t + c_t^m + c_t^x + \frac{2\delta_t}{1-\delta_t}\cdot c_t^x. \tag{19}$$

Multiplying both sides of (19) by P_t and using again (11) yields

$$M_t = \underbrace{P_t\dot{K}_t + P_t c_t^m + P_t c_t^x}_{r_t K_t + w_t L_t^y} + \underbrace{2\delta_t\cdot q_t P_t c_t^x}_{w_t\left(1-L_t^y\right)} \tag{20}$$

where the last term coincides with the exchange plarform's total wage bill by the zero profit condition (15). Result (20) confirms that the CIA constraint imposed on expenditures (7) is equivalent to the assumed legal requirement that all factor incomes must be paid using money, the legal tender.

4 Intertemporal choices and equilibrium notions

4.1 Utility maximizing conditions

The household problem consists of maximizing present-value utility (1) subject to the constraints (6) and (7). The current-value Hamiltonian for the household problem can be written as

$$\begin{aligned} &= \ln\left[\left(c_t^m\right)^\theta\left(c_t^x\right)^{1-\theta}\right] + \lambda_t^K\dot{K}_t + \lambda_t^M\dot{m}_t + \lambda_t^S\left[m_t - \dot{K}_t - c_t^m - q_t\left(1+\delta_t\right)c_t^x\right] = \\ &= \ln\left[\left(c_t^m\right)^\theta\left(c_t^x\right)^{1-\theta}\right] + \lambda_t^K I_t + \lambda_t^M\dot{m}_t + \lambda_t^S\left[m_t - I_t - c_t^m - q_t\left(1+\delta_t\right)c_t^x\right], \end{aligned} \tag{21}$$

where we have defined capital investment as $\dot{K}=I$. This allows us to treat real money m and capital K as state variables while c_t^m, c_t^x and I_t act as control variables; all prices are taken as given under perfect foresight, λ_t^K is the shadow price of capital accumulation, λ_t^M is the shadow price of money accumulation, and λ_t^S is the Khun-Tucker multiplier attached to the CIA constraint. Replacing $\dot{m}_t$ by means of expression (6), and collecting terms for $\dot{K} = I$, the Hamiltonian (21) can be rewritten as

$$\begin{aligned} = \quad & \ln\left[(c_t^m)^\theta (c_t^x)^{1-\theta}\right] + \left(\lambda_t^K - \lambda_t^S - \lambda_t^M\right)\cdot I_t + \\ & +\lambda_t^M\left[\frac{1}{P_t}(w_t + r_t K_t) + v_t - c_t^m - q_t(1+\delta_t)c_t^x - m_t\pi_t\right] + \\ & +\lambda_t^S\cdot\left[m_t - c_t^m - q_t(1+\delta_t)c_t^x\right]. \end{aligned} \tag{22}$$

The necessary conditions for utility maximization are therefore

$$\begin{aligned} c_t^m &= 0 && \rightarrow \frac{\theta}{c_t^m} - \lambda_t^M - \lambda_t^S = 0 \\ c_t^x &= 0 && \rightarrow \frac{1-\theta}{c_t^x} - \left(\lambda_t^M - \lambda_t^S\right) q_t(1+\delta_t) = 0 \\ I_t &= 0 && \rightarrow \lambda_t^K - \lambda_t^S = \lambda_t^M \\ K_t &= \rho\lambda_t^K - \dot{\lambda}_t^K && \rightarrow \rho\lambda_t^K - \dot{\lambda}_t^K = \lambda_t^M \frac{r_t}{P_t} \\ M_t &= \rho\lambda_t^M - \dot{\lambda}_t^M && \rightarrow \rho\lambda_t^M - \dot{\lambda}_t^M = \lambda_t^S - \lambda_t^M \pi_t \end{aligned}$$

along with the transversality conditions

$$\lim_{t\to\infty} \lambda_t^K K_t e^{-\rho t} = 0, \tag{23}$$

$$\lim_{t\to\infty} \lambda_t^M m_t e^{-\rho t} = 0. \tag{24}$$

For future reference, we can rewrite the utility-maximizing conditions as

$$\frac{\theta}{c_t^m} = \lambda_t^K \tag{25}$$

$$\frac{1-\theta}{c_t^x} = \lambda_t^K \cdot q_t(1+\delta_t) \tag{26}$$

$$\lambda_t^K = \lambda_t^M + \lambda_t^S \tag{27}$$

$$\frac{\dot{\lambda}_t^K}{\lambda_t^K} = \rho - \frac{\lambda_t^M}{\lambda_t^K}\cdot\frac{r_t}{P_t} \tag{28}$$

$$\frac{\dot{\lambda}_t^M}{\lambda_t^M} = \rho + \pi_t - \frac{\lambda_t^K - \lambda_t^M}{\lambda_t^M} \tag{29}$$

Equations (25) and (26) imply that the ratio between money-paid and crypto-paid consumption goods is determined by tastes and by the gross real exchange rate between money and crypto-currency:

$$\frac{c_t^m}{c_t^x} = \frac{\theta}{1-\theta}\cdot q_t(1+\delta_t) = \frac{\theta}{1-\theta}\cdot\frac{1+\delta_t}{1-\delta_t} \tag{30}$$

where the last term follows from (11). As intuitive, the share of consumption in money-purchased goods increases with higher crypto-fees.

The co-state equations (28)-(29) can be reduced to a single differential equation by defining the composite multiplier $\lambda_t^R \equiv \lambda_t^M/\lambda_t^K$, which evolves over time according to

$$\frac{\dot{\lambda}_t^R}{\lambda_t^R} = \frac{\dot{\lambda}_t^M}{\lambda_t^M} - \frac{\dot{\lambda}_t^K}{\lambda_t^K} = \pi_t - \frac{\lambda_t^K - \lambda_t^M}{\lambda_t^M} + \frac{\lambda_t^M}{\lambda_t^K} \cdot \frac{r_t}{P_t}$$

that is,

$$\frac{\dot{\lambda}_t^R}{\lambda_t^R} = \pi_t - \frac{1 - \lambda_t^R}{\lambda_t^R} + \lambda_t^R \cdot \frac{r_t}{P_t}. \tag{31}$$

Equation (31) determines the joint dynamics of the shadow values of money and capital and will be used later to determine the properties of long-run equilibria.

4.2 Steady-state and BGP equilibria

The work-horses of dynamic macroeconomics suggest considering two reference notions of long-run equilibria. The first characterizes models of exogenous growth, i.e., Ramsey-like economies where diminishing returns to capital drive down the interest rate over time and imply that, in the absence of productivity growth, consumption and output per capita are stationary in the long run. In the present context, if we assume that the production function of the final sector exhibits diminishing returns to capital (at the firm and at the aggregate level) alongside a constant exogenous level of labor productivity, we obtain a Ramsey-like economy that should, at least in principle, admit a *steady-state equilibrium* in the long run characterized by constant consumption levels. We investigate this point in the first variant of our model, which we label as the 'neoclassical case'.

The second variant of the model is suggested by the endogenous growth literature. In this class of models, the economy's rate of return is sustained in the long run by endogenous forces that eliminate strictly diminishing returns to accumulable factors, implying persistent consumption growth in the long run. In the present context, if we assume that the production function of the final sector incorporates learning-by-doing spillovers through labor productivity – whereby capital exhibits diminishing returns at the firm level but non-diminishing returns at the aggregate level – we obtain a Romer-like economy (Romer (1989)) that should, at least in principle, admit a *balanced growth path equilibrium* delivering sustained growth in consumption and output in the long run. We will refer to this variant of the model as to the 'AK case'.

5 The Neoclassical case

This section describes the general properties of the steady state equilibrium. These properties hold regardless of the specific technology used by the exchange platform and are generally valid for any static CRS production function in goods production (i.e., a linearly homogeneous technology with constant labor productivity: $a_t = a > 0$). In section 8 we will specify technologies for both the exchange platform and the goods sector to derive further results on the impact of technology shocks.

5.1 Consumption and money non-neutrality

Consider an equilibrium with constant consumption. From (25) and (26), stationarity in c_t^m and c_t^x requires a constant multiplier λ_t^K as well as constant crypto-fees,

$$\frac{\mathrm{d}}{\mathrm{d}t} q_t \left(1 + \delta_t\right) = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1 + \delta_t}{1 - \delta_t} = 0,$$

which will be the case for suitable specifications of the technology of the exchange platform. From (28), the steady state $\dot{\lambda}_t^K = 0$ requires that the real rental rate for capital equals the utility dicount rate weighted by the composite multiplier $\lambda_t^R \equiv \lambda_t^M / \lambda_t^K$ previously defined,

$$\frac{r_t}{P_t} = \frac{1}{\lambda_t^R} \cdot \rho. \tag{32}$$

Since ρ is constant and r_t/P_t equals the physical marginal product of capital, a constant real interest rate requires $\dot{\lambda}_t^R = 0$ in (31), which yields

$$\lambda_t^R = \frac{1}{1 + \rho + \pi_t}. \tag{33}$$

By combining (32) with (33), a steady-state equilibrium in the neoclassical case is characterized by the real rate of return

$$\frac{r_t}{P_t} = \rho \cdot \left(1 + \rho + \pi_t\right). \tag{34}$$

Expression (34) shows three important results. First, money is not neutral: a nominal variable – the money inflation rate, π_t – affects real variables in equilibrium – the physical marginal product of capital, r_t/P_t. Second, a neoclassical steady state with constant real interest requires the inflation rate to be constant over time, which in turn imposes a restriction on monetary growth (i.e., a constant money growth rate set by the authority). Third, inflation tends to reduce capital accumulation. In the standard Ramsey model without a CIA constraint on capital purchases, the steady-state condition $r_t/P_t = \rho$ implies a lower interest rate and a higher capital-labor ratio than condition (34) – provided that the money inflation rate is $\pi_t > -\rho$. In other words, unless we observe substantial deflation, the cash-in-advance constraint implies under-accumulation of capital and inefficiently low consumption.

The economic intuition for non-neutrality of money is that the CIA constraint on new capital purchases forces agents to keep money to make real investment but positive inflation increases the real cost of holding money, which affects the real return to investment from the household point of view. This source of non-neutrality does not apply to the crypto-currency – in fact, we have not postulated that cyrpto-currency is necessary to purchase real investment. The crypto-currency is non-neutral for other reasons, namely, the fact that its circulation absorbs real resources (in the form of labor employed in exchange platforms). This point is clarified below.

5.2 Non-neutralities: money versus crypto

In order to assess the role of the crypto-currency, impose the conditions for a neoclassical steady state in the CIA constraint: setting $\dot{K}_t = 0$ in (7), we obtain

$$m_t = c_t^m + \frac{1+\delta_t}{1-\delta_t} c_t^x = \frac{1}{\theta} c_t^m \tag{35}$$

where the last term follows from substituting the utility-maximizing consumption ratio (30). From (35), a steady state in consumption implies a steady state in real money supply, $\dot{m}_t = 0$, which means that the money inflation rate equals the growth rate of money supply. Assuming that the monetary authority lets nominal grow at the constant rate g^M, the inflation rate is constant as well,

$$\pi_t = \dot{M}_t / M_t \equiv g^M. \tag{36}$$

Since π_t only depends on money growth, the dynamics of the supply of crypto-currency do not affect the steady-state condition (34) through this channel: money inflation is independent of crypto inflation. However, the existence of the crypto-market does affect the real interest rate in (34) through a *labor reallocation effect*. In a neoclassical world, the physical marginal product of capital depends on the capital-labor ratio in goods' production, K_t/L_t^y, and L_t^y is in turn affected by the fact that part of the workforce, $L - L_t^y$, is at the same time employed in exchange platforms. Since the crypto-market subtracts resources – in this case, labor inputs – that would have been otherwise used in goods production, the existence of the crypto-currency exherts an additional pressure towards under-accumulation and inefficiently low consumption levels in the steady state. In fact, if all the workforce L could be employed in goods production, condition (34) would be met with an identical capital-labor ratio – say, $K_t'/L = K_t/L_t^y$ – but such ratio would be associated to higher levels of capital and output ($L > L_t^y$ would imply $K_t' > K_t$).

Another effect of the crypto-market is that the fee rate δ_t distorts the relative expenditure shares of money-purchased and crypto-purchased goods, which is immediately evident from (30). In this respect, the extent of the distortion depends on the technology of the exchange platform and on the resulting level of fees. We will present a complete analytical derivation of the balanced growth equilibrium under a specific technology for the exchange platform in section 8.

6 The AK case

Assume that the final good sector comprises an indefinite number of firms exploiting the same technology displaying constant returns to scale at the firm level. Despite diminishing marginal returns to both labor and capital at the firm level, learning-by-doing spillovers at the sectoral level induce constant marginal returns to capital – that is, a constant real interest rate – in the spirit of Romer (1986) and Romer (1989). Assuming identical technologies across firms guarantees a symmetric equilibrium where the economy's final consumption good is produced according to an AK technology.

6.1 Goods production with spillovers

Assume that the final good sector comprises an indefinite number of firms indexed by n. Each firm exploits the production function $y_{n,t} = k_{n,t}{}^{\alpha} \left(\bar{a}_t \ell^y_{n,t}\right)^{1-\alpha}$ where $y_{n,t}$ is output, $k_{n,t}$ is physical capital, $\ell^y_{n,t}$ is labor, $\bar{a}_t$ is workers' productivity, $\alpha \in (0,1)$ is an elasticity parameter. At the firm level, labor productivity $\bar{a}_t$ is taken as given, and profit maximization yields the usual first-order conditions

$$\frac{r_t}{P_t} = \alpha \frac{y_{n,t}}{k_{n,t}}, \tag{37}$$

$$\frac{w_t}{P_t} = (1-\alpha) \frac{y_{n,t}}{\ell^y_{n,t}}, \tag{38}$$

Since firms use identical technologies, the capital-labor ratio is the same in each firm and coincides with the capital-labor ratio at the sectoral level, $k_{n,t}/\ell^y_{n,t} = K_t/L^y_t$. Assume learning-by-doing spillovers at the sectoral level whereby the use of capital increases workers' productivity. We postulate the spillover function $\bar{a} = A^{\frac{1}{1-\alpha}} (K_t/L^y_t)$, which implies that the productivity of each worker increases with the capital intensity of the sector. The intuition is that capital use induces complementary efficiency gains: each worker uses machines and a more intense use of machines in the sector makes each unit of labor more efficient. Substituting the spillover function in firms' technologies, sectoral output becomes linear in sectoral capital,

$$Y_t = AK_t, \tag{39}$$

like in standard growth models *á la* Romer (1989). Consequently, the equilibrium interest and wage rates are

$$\frac{r_t}{P_t} = \alpha A, \tag{40}$$

$$\frac{w_t}{P_t} = (1-\alpha) \cdot A \cdot (K_t/L^y_t). \tag{41}$$

The fact that the real return to capital αA is constant creates the possibility of balanced growth paths (BGPs), that is, scenarios in which the economy exhibits sustained endogenous growth in the long run. Given the non-neutrality of money and crypto-currency, the natural question is whether nominal variables will affect not only income levels but also income growth in the long run. The next subsection tackles this issue in general terms without assuming a specific technology for the exchange platform.

6.2 Balanced growth equilibrium: general properties

Consider a balanced growth equilibrium where consumption levels of both goods grow at the constant rate

$$g^C = \frac{\dot{c}^m_t}{c^m_t} = \frac{\dot{c}^x_t}{c^x_t}$$

and there is a constant rate of crypto-fees, $\dot{\delta}_t = 0$, which will be the case for suitable specifications of the technology of the exchange platform. From (28), a constant growth rate $-\dot{\lambda}_t^K/\lambda_t^K = g^C > 0$ requires

$$g^C = \frac{\lambda_t^M}{\lambda_t^K} \cdot \alpha A - \rho > 0 \tag{42}$$

where we have substituted the interest rate $r_t/P_t = \alpha A$ from (40). From (42), balanced growth requires $\lambda_t^R \equiv \lambda_t^M/\lambda_t^K$ to be constant as well: imposing $\dot{\lambda}_t^R = 0$ in (31) yields the second-order equation

$$\alpha A \cdot \left(\lambda_t^R\right)^2 + \lambda_t^R \left(1 + \pi_t\right) - 1 = 0$$

with positive root given by

$$\lambda_t^R = \frac{\sqrt{\left(1 + \pi_t\right)^2 + 4\alpha A} - \left(1 + \pi_t\right)}{2\alpha A}. \tag{43}$$

Substituting this result into (42), the balanced growth rate equals

$$g^C \equiv \frac{\dot{c}_t^m}{c_t^m} = \frac{\dot{c}_t^x}{c_t^x} = \frac{\sqrt{\left(1 + \pi_t\right)^2 + 4\alpha A} - \left(1 + \pi_t\right)}{2} - \rho \tag{44}$$

Result (44) shows that money is neither neutral nor superneutral: the inflation rate affects real growth in a BGP equilibrium. In particular, the derivative of the balanced growth rate with respect to π_t equals

$$\frac{\partial g^C}{\partial \pi_t} = -\frac{4\alpha A + 2\pi_t + \pi_t^2}{2\left(1 + \pi_t\right)^2 + 8\alpha A} \tag{45}$$

and is strictly negative for any positive (or even negative, but relatively small) rate of money inflation. That is, positive inflation slows down real growth in this model. The reason is, conceptually, the same as that in the neoclassical case: the CIA constraint on new capital purchases forces agents to keep money to make real investment but positive inflation increases the real cost of holding money, which affects the real return to investment from the household point of view. Differently from the neoclassical case, where the interest rate determines the *stationary level* of consumption in the steady state, in the AK variant of the model the interest rate determines the *growth rate* of consumption along the balanced growth path. Therefore, in the AK case, the negative effect money inflation on the real return to investment translates into a negative effect on the economy's growth rate.

The transmission from monetary policy to inflation can be addressed by imposing the conditions for a BGP equilibrium in the CIA constraint. Setting $\dot{K}_t = g^C K_t$ in (7), we obtain

$$m_t = g^C K_t + c_t^m + \frac{1 + \delta_t}{1 - \delta_t} c_t^x = g^C K_t + \frac{1}{\theta} c_t^m \tag{46}$$

where the last term follows from substituting the utility-maximizing consumption ratio (30). Since a BGP requires capital and consumption to grow at rate g^C, the ratio c_t^m/K_t must be constant:

denoting this (endogenous) variable as $\chi_{CD}^{m} \equiv c_t^m / K_t$ we can rewrite (46) as[1]

$$m_t = \left(g^C + \frac{\chi_{CD}^m}{\theta} \right) \cdot K_t. \tag{47}$$

Equation (47) implies that a constant growth rate g^C requires that real money supply grows over time at the same constant rate, $\dot{m}_t / m_t = \dot{K}_t / K_t = g^C$. This in turn means that a constant growth rule for nominal money supply, $\dot{M}_t / M_t = g^M$, will imply a constant inflation rate π and a constant real growth rate for the economy g^C that satisfies the BGP relation

$$g^M = \pi + g^C. \tag{48}$$

Using (44) to substitute g^C in (48) and rearranging terms yields

$$\sqrt{(1+\pi_t)^2 + 4\alpha A} - (1 - \pi_t) = 2 \left(g^M + \rho \right) \tag{49}$$

The above results obey a precise causality: given the exogenous monetary rule set by authorities, the growth rate of money supply g^M determines inflation π according to (49). The inflation rate π then determines the economy's real growth rate g^C according to equation (44).

Since money inflation only depends on nominal money growth, the dynamics of the supply of crypto-currency do not affect real growth through this channel: money inflation, $\pi_t = \dot{P}_t / P_t$, is independent of crypto inflation, $\pi_t^* = \dot{P}_t^* / P_t^*$. The main consequence of the crypto-currency is a permanent change in the level of the real wage induced by a *labor reallocation effect*. Expression (41) implies that in a BGP equilibrium – where capital grows at rate $\bar{g}_t$ while employment levels L_t^y and $L - L_t^y$ are stationary – the real wage will grow at the balanced rate $\bar{g}_t$ while sectoral employment determines a permanent level effect: the higher the employment in the exchange platform $L - L_t^y$, the lower the *levels* of the equilibrium real wage $w_t / P_t = (1/L_t^y) \cdot (1 - \alpha) A K_t$ along the BGP. We will present a complete analytical derivation of the balanced growth equilibrium under a specific technology for the exchange platform in section 8.

7 Substitutability and money-crypto interactions

In this section, we extend the model to replace Cobb-Douglas perferences with a CES utility function. The next section shows how the relevant dynamic system changes when money-purchased and crypto-purchased goods are allowed to be strict complements or strict substitutes. The subsequent sections derive general results for neoclassical steady-state equilibria and for BGP equilibria with endogenous growth in the same vein as the previous sections.

7.1 Intertemporal choices under CES preferences

Suppose that the instantaneous utility function $u(c_t^m, c_t^x)$ in (1) is replaced by the CES form

$$u(c_t^m, c_t^x) = \left[\theta \cdot (c_t^m)^{\frac{\sigma-1}{\sigma}} + (1-\theta) \cdot (c_t^x)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \tag{50}$$

[1] Subsection 8.4 includes a complete derivation of the equilibrium value of χ_{CD}^m.

where $\sigma > 0$ is the elasticity of substitution between money-purchased and crypto-purchased goods. When $\sigma < 1$, the consumer perceives the two types of consumption as strict complements. When $\sigma > 1$, the consumer perceives the two types of consumption as strict substitutes. Letting $\sigma \to 1$, the utility function reduces to the Cobb-Douglas form $u(c_t^m, c_t^x) = (c_t^m)^\theta (c_t^x)^{1-\theta}$ assumed before. In this modified model, the household maximizes intertemporal utility

$$U \equiv \int_0^\infty e^{-\rho t} \ln [u(c_t^m, c_t^x)]\, dt$$

subject to the same dynamic constraints considered before. Proceeding with the same steps shown in section 4, we obtain a system of utility-maximizing conditions in which the marginal utilities are not separable: both $\partial u/\partial c_t^m$ and $\partial u/\partial c_t^x$ depend on money-purchased *and* on crypto-purchased quantities, c_t^m and c_t^x. More precisely, system (25)-(29) is replaced by

$$\frac{1}{u(c_t^m, c_t^x)} \cdot \frac{\partial u}{\partial c_t^m} = \lambda_t^K \tag{51}$$

$$\frac{1}{u(c_t^m, c_t^x)} \cdot \frac{\partial u}{\partial c_t^x} = \lambda_t^K \cdot q_t (1 + \delta_t) \tag{52}$$

$$\lambda_t^K = \lambda_t^M + \lambda_t^S \tag{53}$$

$$\frac{\dot{\lambda}_t^K}{\lambda_t^K} = \rho - \frac{\lambda_t^M}{\lambda_t^K} \cdot \frac{r_t}{P_t} \tag{54}$$

$$\frac{\dot{\lambda}_t^M}{\lambda_t^M} = \rho + \pi_t - \frac{\lambda_t^K - \lambda_t^M}{\lambda_t^M} \tag{55}$$

The key difference with respect to the model with Cobb-Douglas utility is that relative expenditure shares now depend on relative prices. By combining (51) with (52), we obtain the utility-maximizing condition

$$\frac{\partial u}{\partial c_t^m} \cdot q_t (1 + \delta_t) = \frac{\partial u}{\partial c_t^x},$$

where we can substitute the marginal utilities calculated from (50),

$$\frac{\partial u}{\partial c_t^m} = \theta \cdot \left(\frac{u(c_t^m, c_t^x)}{c_t^m} \right)^{\frac{1}{\sigma}} \quad \text{and} \quad \frac{\partial u}{\partial c_t^x} = (1 - \theta) \cdot \left(\frac{u(c_t^m, c_t^x)}{c_t^x} \right)^{\frac{1}{\sigma}}, \tag{56}$$

obtaining the consumption ratio

$$\frac{c_t^m}{c_t^x} = \left[\frac{\theta}{1 - \theta} \cdot q_t (1 + \delta_t) \right]^\sigma. \tag{57}$$

Dividing both sides by $q_t (1 + \delta_t)$ yields the real expenditure ratio, i.e., the expenditure on money-purchased goods relative to that on crypto-purchased goods,

$$\frac{c_t^m}{q_t (1 + \delta_t) c_t^x} = \left(\frac{\theta}{1 - \theta} \right)^\sigma \cdot \left(\frac{1 - \delta_t}{1 + \delta_t} \right)^{1-\sigma}, \tag{58}$$

where we have used (11) to eliminate the real exchange rate on the right hand side. Expression (58) shows that a change in the crypto-fee has generally ambiguous effects on relative expenditures.

If the consumer perceives money-purchased and crypto-purchased goods as complements, $\sigma < 1$, an increase in the fee rate δ_t prompts them to reduce the left hand side of (58) – that is, to reduce relative spending on money-purchased goods to spend a higher fraction of consumption expenditure on crypto-purchased goods. Viceversa, if the household perceives money-purchased and crypto-purchased goods as substitutes, $\sigma > 1$, an increase in the fee rate δ_t prompts them to reduce the expenditure share on crypto-purchased goods. Armed with this result, we can now investigate the general properties of neoclassical steady-state equilibria and of BGP equilibria under CES preferences.

7.2 Neoclassical steady state with CES preferences

In this subsection, we consider a neoclassical steady state equilibrium with constant consumption. The first part of the analysis is very similar to that in section 5, with small differences that we emphasize below. The distortions induced by the crypto-market become more evident in the complete analytical solution, which clarifies reallocation effects and their consequences for the steady-state capital stock.

From (51) and (52), stationarity in c_t^m and c_t^x requires a constant multiplier λ_t^K as well as constant crypto-fees,

$$\frac{\mathrm{d}}{\mathrm{d}t}q_t\left(1+\delta_t\right) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1+\delta_t}{1-\delta_t} = 0,$$

which will be the case for suitable specifications of the technology of the exchange platform. From (54), the steady state $\dot{\lambda}_t^K = 0$ requires that the real rental rate for capital equals the utility dicount rate weighted by the composite multiplier $\lambda_t^R \equiv \lambda_t^M/\lambda_t^K$ previously defined: $r_t/P_t = \rho/\lambda_t^R$. Since ρ is constant and r_t/P_t equals the physical marginal product of capital, a constant real interest rate requires $\dot{\lambda}_t^R = 0$ in (31), which yields $\lambda_t^R = 1/\left(1+\rho+\pi_t\right)$ and, hence, a steady-state real rate of return

$$\frac{r_t}{P_t} = \rho \cdot \left(1+\rho+\pi_t\right).$$

As noted before, (i) money is not neutral and (ii) inflation tends to reduce capital accumulation. In order to assess the role of the crypto-currency, impose the conditions for a neoclassical steady state in the CIA constraint: setting $\dot{K}_t = 0$ in (18), we obtain

$$m_t = c_t^m \cdot \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot \left(\frac{1+\delta_t}{1-\delta_t}\right)^{1-\sigma}\right] \tag{59}$$

where the last term follows from substituting the consumption expenditures ratio (58). Since $\dot{\delta}_t = 0$ by construction of the steady state, result (59) implies that stationary consumption is associated with $\dot{m}_t = 0$, that is, the money inflation rate equals the growth rate of money supply set by the authority, $\pi_t = \dot{M}_t/M_t$. Since π_t only depends on money growth, the dynamics of the supply of crypto-currency do not affect the steady-state condition (34) through this channel: money inflation equals $\pi_t = \dot{M}_t/M_t$ and is therefore independent of crypto-currency supply. This conclusion also holds with Cobb-Douglas preferences, as shown in subsection 5. However, differently from the model with Cobb-Douglas preferences, the degree of substitutability between money-purchased and crypto-purchased goods affects the price level. The right hand side of (59) shows that when consumers perceive money-purchased and crypto-purchased goods as strict complements (substitutes), a higher fee tends to increase (reduce) the equilibrium real money supply at given consumption levels. The reason is that under complementarity (substitutability), higher fees prompt consumers to spend relatively more on crypto-purchased goods, exherting a downward pressure on the relative price of money-purchased goods and, hence, an upward pressure on the equilibrium real money supply at given consumption levels.

Besides the effect on price levels, it should be remembered that the crypto-currency is not neutral because, as shown in subsection 5, it affects the real interest rate in (34) through a *labor reallocation effect*: the physical marginal product of capital depends on the capital-labor ratio in goods' production, K_t/L_t^y, and L_t^y is in turn affected by employment in exchange platforms via the labor market.

7.3 Balanced growth equilibrium with CES preferences

In this subsection, we consider a BGP equilibrium with sustained growth generated by the technology described in subsection 6.1: sectoral spillovers induce linear returns to capital at the aggregate level, $Y_t = AK_t$, with real factor rewards given by $r_t/P_t = \alpha A$ and $w_t/P_t = (1-\alpha)\cdot A \cdot (K_t/L_t^y)$.

The general properties of the BGP are as follows. Consider a balanced growth equilibrium where consumption levels of both goods grow at the constant rate $g^C = \dot{c}_t^m/c_t^m = \dot{c}_t^x/c_t^x$, and there is a constant rate of crypto-fees, $\dot{\delta}_t = 0$, which will be the case for suitable specifications of the technology of the exchange platform. Time-differentiating (51) and (52) with $\dot{\delta}_t = 0$ we obtain

$$\frac{\dot{\lambda}_t^K}{\lambda_t^K} = \frac{1}{\frac{\partial u}{\partial c_t^m}} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial u}{\partial c_t^m} - \frac{\dot{u}\left(c_t^m, c_t^x\right)}{u\left(c_t^m, c_t^x\right)}, \tag{60}$$

$$\frac{\dot{\lambda}_t^K}{\lambda_t^K} = \frac{1}{\frac{\partial u}{\partial c_t^x}} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial u}{\partial c_t^x} - \frac{\dot{u}\left(c_t^m, c_t^x\right)}{u\left(c_t^m, c_t^x\right)}. \tag{61}$$

From (50), the growth rate of utility equals

$$\frac{\dot{u}\left(c_t^m, c_t^x\right)}{u\left(c_t^m, c_t^x\right)} = \frac{\theta \cdot \left(c_t^m\right)^{\frac{\sigma-1}{\sigma}} g^C + (1-\theta)\cdot\left(c_t^x\right)^{\frac{\sigma-1}{\sigma}} g^C}{\theta \cdot \left(c_t^m\right)^{\frac{\sigma-1}{\sigma}} + (1-\theta)\cdot\left(c_t^x\right)^{\frac{\sigma-1}{\sigma}}} = g^C. \tag{62}$$

From (56), the growth rate of marginal utility for either good reads

$$\frac{1}{\frac{\partial u}{\partial c_t^{\cdot}}} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial u}{\partial c_t^{\cdot}} = \frac{1}{\sigma}\left[\frac{\dot{u}\left(c_t^m, c_t^x\right)}{u\left(c_t^m, c_t^x\right)} - g^C\right] = 0 \tag{63}$$

Substituting results (62)-(63) in either (60) or (61) yields $\dot{\lambda}_t^K/\lambda_t^K = -g^C$. Hence, from (54) and the constant interest rate $r_t/P_t = \alpha A$, we have

$$g^C = \frac{\lambda_t^M}{\lambda_t^K} \cdot \alpha A - \rho \tag{64}$$

which implies a constant composite multiplier $\lambda_t^R = \lambda_t^M/\lambda_t^K$. Setting $\dot{\lambda}_t^R = 0$ in (31) yields again result (43) and thereby the balanced growth rate

$$g^C \equiv \frac{\dot{c}_t^m}{c_t^m} = \frac{\dot{c}_t^x}{c_t^x} = \frac{\sqrt{(1+\pi_t)^2 + 4\alpha A} - (1+\pi_t)}{2} - \rho. \tag{65}$$

As noted before, any positive (or even negative, but relatively small) rate of money inflation reduces g^C because money inflation increases the cost of holding money – and holding money is necessary to have the liquidity needed to make real investment. Result (65) shows that the balanced growth rate under CES preferences is the same as in the Cobb-Douglas case with $\sigma = 1$. However, the current hypothesis that goods can be perceived as complements modifies the impact of monetary policy on the general price level. To see this formally, use result (58) to write real consumption expenditures as

$$c_t^m + q_t(1+\delta_t)\,c_t^x = c_t^m \cdot \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot \left(\frac{1+\delta_t}{1-\delta_t}\right)^{1-\sigma}\right] \tag{66}$$

and impose the conditions for a BGP equilibrium in the CIA constraint: setting $\dot{K}_t = g^C K_t$ in (7), we obtain

$$m_t = \bar{g}_t K_t + c_t^m + q_t(1+\delta_t)\,c_t^x = \bar{g}_t K_t + c_t^m \cdot \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot \left(\frac{1+\delta_t}{1-\delta_t}\right)^{1-\sigma}\right] \tag{67}$$

where the last term follows from (66). Since a BGP requires capital and consumption to grow at rate g^C, the ratio c_t^m/K_t must be constant: denoting this (endogenous) variable as $\chi^m \equiv c_t^m/K_t$ we can rewrite (67) as[2]

$$m_t = \left\{g^C + \chi^m \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot \left(\frac{1+\delta_t}{1-\delta_t}\right)^{1-\sigma}\right]\right\} \cdot K_t. \tag{68}$$

Equation (68) implies that a constant growth rate g^C requires that real money supply grows over time at the same constant rate, $\dot{m}_t/m_t = \dot{K}_t/K_t = g^C$. This in turn means that a constant growth rule for nominal money supply, $\dot{M}_t/M_t = g^M$, will imply a constant inflation rate π and a constant real growth rate for the economy g^C that satisfies the BGP relation $g^M = \pi + g^C$. As shown before (cf. equation (49) in the previous section), the growth rate of money supply g^M determines inflation π according to

$$\sqrt{(1+\pi)^2 + 4\alpha A} - (1-\pi) = 2\left(g^M + \rho\right),$$

and the inflation rate π then determines the economy's real growth rate g^C according to equation (65). The novel result contained in (68) is that the degree of substitutability between money-purchased and crypto-purchased goods directly affects the whole time path of the price level. Substituing $m_t = M_t/P_t$ in (68) and rearranging terms, we obtain

$$P_t = \frac{1}{g^C + \chi^m \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot \left(\frac{1+\delta_t}{1-\delta_t}\right)^{1-\sigma}\right]} \cdot \frac{M_t}{K_t}. \tag{69}$$

[2] Subsection 8.4 includes a complete derivation of the equilibrium value of χ^m.

Result (69) implies that crypto-fees permanently reduce (increase) the money price level when consumers perceive money-purchased and crypto-purchased goods as complements (substitutes). In particular, if the monetary authority sets a constant money growth rule from time zero onwards and the crypto-fee rate is constant from time zero onwards, the whole time path of the price level is given by

$$P_t = \frac{1}{g^C + \chi^m \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot \left(\frac{1+\delta}{1-\delta}\right)^{1-\sigma} \right]} \cdot \frac{M_0}{K_0} \cdot e^{\pi t}. \tag{70}$$

Expression (70) shows that, for a given chosen monetary policy rule g^M, which determines real growth g^C and the inflation rate π, the elasticity of substitution σ and the crypto-fee rate δ determine how high or low the initial price level P_0, and thereby all subsequent price levels, will be. Under complementarity, $\sigma < 1$, a higher δ yields a lower price level because higher crypto-fees prompt consumers to reduce their relative demand for money-purchased goods. Under substitutability, $\sigma > 1$, a higher δ yields a higher price level because higher crypto-fees prompt consumers to increase their relative demand for money-purchased goods. Since δ is positively related to the real exchange rate – see equation (11) – it follows that a real appreciation of the crypto-currency induced by higher crypto-fees affects the price level of money-purchased goods permanently and in opposite directions depending on the value of the elasticity of substitution σ.

As we have shown in section 6.2, the crypto-market permanently affects real wage levels via a *labor reallocation effect*: the higher the employment in the exchange platform $L - L_t^y$, the lower the *levels* of the equilibrium real wage $w_t/P_t = (1/L_t^y) \cdot (1-\alpha)\, AK_t$ along the BGP. This result is obviously confirmed in the model with CES preferences.

8 Complete derivations and shocks

This section presents full analytical derivations of (i) the Neoclassical steady state and (ii) the BGP equilibrium for the extended model with CES preferences, which allows us to derive more general results (the predictions for the model with Cobb-Douglas preferences can be obtained as a special case by setting $\sigma = 1$). For each variant of the model, we obtain a reduced system of equilibrium relationships that determines all endogenous variables and allows us to investigate the effect of exogenous shocks.

8.1 Exchange platform: specifics

Assume that the exchange platform is a competitive sector with free entry of 'exchange firms'. Each firm n hires $\ell_{n,t}^x$ workers to perform currency exchange operations according to a linear technology: each worker's cost to the firm is proportional to the monetary value of the transaction, with proportionality factor $\xi > 0$,

$$w_t = \xi \cdot \eta_{j,t}, \tag{71}$$

where w_t is the wage rate prevailing in the labor market and $\eta_{j,t}$ is the money value of the transaction performed by agent j. Total employment in the currency exchange sector, $L_t^x = \sum_n \ell_{n,t}^x = 1 - L_t^y$, satisfies the demand for currency conversion. Therefore, the sectoral wage bill reads

$$w_t \cdot (1 - L_t^y) = \xi \cdot \int_0^{L_t^x} \eta_{j,t} dj = \xi \cdot (P_t c_t^x + P_t c_t^x) \,, \tag{72}$$

where the last term on the right hand side is the market clearing condition for exchange services whereby the money value of total transactions includes those (i) requested by consumers purchasing c_t^x and those (ii) requested by firms selling c_t^x. Exchange firms take the exchange rate as given and set the crypto-fee rate in Bertrand competition. The resulting zero-profit condition, as shown in subsection 3.3, is $2\delta_t Q_t P_t^* c_t^x = w_t \,(1 - L_t^y)$ and can be rewritten in real terms as

$$2\delta_t q_t c_t^x = \frac{w_t}{P_t} \cdot (1 - L_t^y) \,. \tag{73}$$

From (72) and (73), it follows that $\delta_t q_t = \xi$. Combining this result with the real exchange rate in (11), the crypto-fee rate associated with zero profits in the exchange platform reads

$$\delta_t = \frac{\xi}{1+\xi} \equiv \delta \tag{74}$$

which is constant over time. We now have all the elements to derive anlytically the neoclassical steady state equilibrium and the BGP equilibrium in the AK model.

8.2 Neoclassical steady state: full derivation

In the neoclassical case, we normalize labor productivity $a_t = 1$ and assume a Cobb-Douglas production function $Y_t = (K_t)^\alpha \, (L_t^y)^{1-\alpha}$ for the final setor. The profit-maximizing conditions yield the demand schedules for capital and labor,

$$\frac{r_t}{P_t} = \alpha \cdot \left(\frac{L_t^y}{K_t}\right)^{1-\alpha} , \tag{75}$$

$$\frac{w_t}{P_t} = (1-\alpha) \cdot \left(\frac{K_t}{L_t^y}\right)^{\alpha} . \tag{76}$$

Combining (75) with the steady-state condition for the interest rate (34) and steady-state inflation rate $\pi_t = g^M$ from (36) yields the capital-labor ratio for the final sector in the neoclassical steady state,

$$\frac{K_t}{L_t^y} = \left[\frac{\alpha}{\rho \cdot (1 + \rho + g^M)}\right]^{\frac{1}{1-\alpha}} . \tag{77}$$

From the zero-profit condition in the exchange platform (73) and the equilibrium fee rate (74), labor demand by currency-exchange firms is

$$\frac{w_t}{P_t} = 2\frac{\delta_t q_t}{1 - L_t^y} c_t^x = \frac{2\xi}{1 - L_t^y} c_t^x . \tag{78}$$

The equilibrium in the labor market is characterized by real wage equalization which, from (76) and (78), implies

$$\frac{K_t}{L_t^y} = \left[\frac{2\xi}{1-\alpha} \cdot \frac{c_t^x}{1-L_t^y}\right]^{\frac{1}{\alpha}}. \tag{79}$$

Using (11) and (74), the ratio between money-purchased and crypto-purchased goods (57) equals

$$\frac{c_t^m}{c_t^x} = \left[\frac{\theta\,(1+2\xi)}{1-\theta}\right]^{\sigma}. \tag{80}$$

Using (74), the steady-state level of real money supply (35) equals

$$m_t = c_t^m \cdot \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot (1+2\xi)^{1-\sigma}\right]. \tag{81}$$

The goods' market clearing condition (8) in the steady state implies

$$c_t^x = (K_t)^{\alpha}\,(L_t^y)^{1-\alpha} - c_t^m. \tag{82}$$

Reduced system (neoclassical steady state). Equations (77), (79), (80), (81) and (82) form a *reduced equilibrium system* that allows us to determine the steady state values of inputs and consumption levels – and thereby all the related endogenous variables of interest – in the neoclassical steady state:

$$\frac{K_{ss}}{L_{ss}^y} = \left[\frac{\alpha}{\rho \cdot (1+\rho+g^M)}\right]^{\frac{1}{1-\alpha}} \tag{83}$$

$$\frac{K_{ss}}{L_{ss}^y} = \left[\frac{2\xi}{1-\alpha} \cdot \frac{c_{ss}^x}{1-L_{ss}^y}\right]^{\frac{1}{\alpha}} \tag{84}$$

$$\frac{c_{ss}^m}{c_{ss}^x} = \left[\frac{\theta\,(1+2\xi)}{1-\theta}\right]^{\sigma} \tag{85}$$

$$m_{ss} = \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot (1+2\xi)^{1-\sigma}\right] \cdot c_{ss}^m \tag{86}$$

$$c_{ss}^x = (K_{ss})^{\alpha}\,(L_{ss}^y)^{1-\alpha} - c_{ss}^m \tag{87}$$

The reduced system (83)-(87) comprises five equations determining five unknowns: capital K_{ss}, labor employed in the final sector L_{ss}^y, consumption of money-purchased goods c_{ss}^m, consumption of crypto-purchased goods c_{ss}^x, and real money holdings m_{ss}. The exogenous parameters reflect technologies (α, ξ), preferences (θ, σ, ρ) and the monetary policy rule set by the authority, $\dot{M}_t/M_t = g^M$. The equilibrium values $(K_{ss}, L_{ss}^y, c_{ss}^m, c_{ss}^x, m_{ss})$ allow us to calculate real factor prices r_t/P_t and w_t/P_t from (75)-(76), the crypto-fee rate from (74), the real exchange rate from (11), and steady-state utility $u\left(c_t^m, c_t^x\right)$ from (50). The next subsection presents some numerical results describing the effects of exogenous shocks.

8.3 Neoclassical steady state: numerical analysis

In this subsection, we introduce a numerical illustration of the neoclassical steady state and study cases of strict complementarity, strict substitutability, and Cobb-Douglas preferences. We then proceed to assess the effects of exogenous changes in the growth rate of nominal money, in the crypto-fee rate (due to an exogenous rise in ξ), and in the taste parameter θ on the endogenous variables in the reduced system above. Parameter values are reported in Table 1 along with the equilibrium level of the endogenous variables in Table 2.

Table 1: Parameter values.

Preferences	**Technology**	**Monetary policy rule**
$\theta = 0.3$	$\alpha = 0.3$	$g^M = 0.045$
$\rho = 0.02$	$\xi = 0.05$	

Table 2: Benchmark results.

	K_{ss}	L^y_{ss}	c^m_{ss}	c^x_{ss}	$\frac{r}{p}$	$\frac{w}{p}$	ϕ	q	$u\left(c^m_t, c^x_t\right)$
$\sigma = 0.5$	40.3413	0.9219	1.1660	1.6982	0.0213	2.1748	0.0476	1.050	1.4937
$\sigma = 1$	39.8858	0.9115	0.9073	1.9246	0.0213	2.1748	0.0476	1.05	-
$\sigma = 1.5$	39.4957	0.9026	0.6857	2.1185	0.0213	2.1748	0.0476	1.050	1.5754

8.3.1 Neoclassical shock analysis

An increase in g^M (faster monetary growth). A 10% increase in the money supply leads to monetary non-neutrality, reflected in a decline in capital stock (K_{ss}), overall consumption (c^m_{ss} and c^x_{ss}) , and real wages ($\frac{w}{p}$) across all substitution levels (σ) in Table 3. The reduction in capital investment is driven by inflation eroding real savings, while consumption decreases due to reduced purchasing power. Utility declines more sharply when money and crypto-currencies are substitutes ($\sigma = 1.5$) because consumers shift more heavily toward crypto-currencies, amplifying the negative impact of rising transaction costs. The rental rate of capital ($\frac{r}{p}$) increases due to reduced capital availability, while crypto-fees (ϕ) and the real exchange rate (q) remain unchanged. This monetary non-neutrality arises from inflationary pressure, negatively impacting the economy's key variables and altering the allocation of resources between sectors.

An increase in ξ (which raises fees, δ). As shown in Table 4, a 10% increase in ξ raises crypto-currency transaction costs, leading to a decline in capital stock (K_{ss}), labor in the goods sector (L^y_{ss}), and consumption of both money (c^m_{ss}) and crypto-purchased goods (c^x_{ss}) across all σ levels, except for a small increase in c^m_{ss} when $\sigma = 1.5$ as consumers shift away from crypto-purchased goods. Real wages remain unchanged, but the crypto-fee (ϕ) and real exchange rate (q) rise, reflecting higher transaction costs. Utility falls due to reduced consumption, with the largest impact seen when fiat money and crypto-currency are substitutes ($\sigma = 1.5$).

Table 3: Shock analysis (10% increase in the money supply).

	K_{ss}	L_{ss}^y	c_{ss}^m	c_{ss}^x	$\frac{r}{p}$	$\frac{w}{p}$	ϕ	q	$u\left(c_t^m, c_t^x\right)$
$\sigma = 0.5$	40.0990↓	0.9219	1.1639↓	1.6952↓	0.0214↑	2.1709↓	0.0476	1.050	1.4910↓
$\sigma = 1$	39.6463↓	0.9115	0.9057↓	1.9211↓	0.0214↑	2.1709↓	0.0476	1.050	-
$\sigma = 1.5$	39.2585↓	0.9026	0.6845↓	2.1146↓	0.0214↑	2.1709↓	0.0476	1.050	1.5725↓

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 2. The absence of an arrow signifies no change compared to the benchmark.

Table 4: Shock analysis (10% increase in the fee structure).

	K_{ss}	L_{ss}^y	c_{ss}^m	c_{ss}^x	$\frac{r}{p}$	$\frac{w}{p}$	ϕ	q	$u\left(c_t^m, c_t^x\right)$
$\sigma = 0.5$	40.0350↓	0.9149↓	1.1603↓	1.6822↓	0.0213	2.1748	0.0521↑	1.055↑	1.4822↓
$\sigma = 1$	39.5470↓	0.9038↓	0.9051↓	1.9027↓	0.0213	2.1748	0.0521↑	1.0550↑	-
$\sigma = 1.5$	39.1285↓	0.8942↓	0.6863↑	2.0918↓	0.0213	2.1748	0.0521↑	1.055↑	1.5603↓

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 2. The absence of an arrow signifies no change compared to the benchmark.

A reduction in θ (higher taste for crypto-purchased goods). A 10% decrease in θ leads to an increase in consumption of crypto goods (c_{ss}^x) across all σ levels in Table 5. This shift reduces the consumption of money-purchased goods (c_{ss}^m) and decreases both capital stock (K_{ss}) and labor allocated to the goods sector (L_{ss}^y). Utility ($u(c_t^m, c_t^x)$) increases due to the higher consumption of crypto goods, with the most pronounced increase seen when money and crypto are substitutes ($\sigma = 1.5$). Real wages ($\frac{w}{p}$) and the crypto fee (ϕ) remain unchanged, while the rate of return on capital ($\frac{r}{p}$) experiences a slight decrease when $\sigma = 1.5$. This suggests that a stronger preference for crypto-purchased goods and a reallocation of resources towards crypto-based consumption, affecting production and investment patterns in the economy.

Table 5: Shock analysis: 10% decrease in θ (higher taste for crypto-purchased goods).

	K_{ss}	L_{ss}^y	c_{ss}^m	c_{ss}^x	$\frac{r}{p}$	$\frac{w}{p}$	ϕ	q	$u\left(c_t^m, c_t^x\right)$
$\sigma = 0.5$	40.2477↓	0.9198↓	1.1129↓	1.7447↑	0.0213	2.1748	0.0476	1.050	1.5128↑
$\sigma = 1$	39.7244↓	0.9078↓	0.8156↓	2.0048↑	0.0213	2.1748	0.0476	1.050	-
$\sigma = 1.5$	39.3006↓	0.8981↓	0.5749↓	2.2154↑	0.0213↓	2.1748	0.0476	1.050	1.6270↑

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 2. The absence of an arrow signifies no change compared to the benchmark.

8.4 BGP equilibrium: full derivation

As shown in subsection 6.1, final output in the AK model equals $Y_t = AK_t$ and the real rental rate for capital is $r_t/P_t = \alpha A$. From (41), labor demand in the final sector implies a real wage $w_t/P_t = (1-\alpha) \cdot A \cdot (K_t/L_t^y)$, whereas, irrespective of the final sector's technology, labor demand in the exchange platform is given by (78). Therefore, wage equalization in the labor market implies

$$\frac{L_t^y}{1-L_t^y} = \frac{A(1-\alpha)}{2\xi} \cdot \frac{K_t}{c_t^x}. \tag{88}$$

Since the crypto-fee rate $\delta_t = \xi/(1+\xi)$ is constant over time and the monetary authority is assumed to follow a constant money growth rule $\dot{M}_t/M_t = g^M$, the AK model admits a permanent BGP equilibrium such that the economy exhibits a constant growth rate from time zero onwards. This implies that, differently from the neoclassical case where we focus on steady-state results – the AK model allows us to build a reduced equilibrium system determining the entire time path of the economy. The key relationship to derive is the equilibrium ratio of consumption to capital which, in this class of models, is a jump variable that settles in its only permanent feasible steady state from time zero onwards. From $Y_t = AK_t$ and (8), the growth rate of capital obeys

$$\frac{\dot{K}_t}{K_t} = A - \frac{c_t^m + c_t^x}{K_t} = A - \frac{C_t}{K_t}. \tag{89}$$

From (65), the growth rate of consumption equals

$$\frac{\dot{C}_t}{C_t} = \frac{\sqrt{(1+\pi_t)^2 + 4\alpha A} - (1+\pi_t)}{2} - \rho \equiv g^C. \tag{90}$$

The above expressions imply that, defining $\chi_t \equiv C_t/K_t$, the growth rate of the consumption-capital ratio obeys

$$\frac{\dot{\chi}_t}{\chi_t} = g^C - A + \chi_t, \tag{91}$$

which is a dynamically unstable equation whose uniques steady state is

$$\chi_* = A - g^C = A + \rho - \frac{\sqrt{(1+\pi_t)^2 + 4\alpha A} - (1+\pi_t)}{2}. \tag{92}$$

It can be shown by standard arguments that setting $\chi_t = \chi_*$ in each $t \in [0, \infty)$ is the only solution that is compatible with (i) the conditions for intertemporal utility maximization and with (ii) satisfying the capital accumulation constraint along the entire time path.[3] Therefore, the BGP equilibrium is characterized by a constant consumption-capital ratio from time zero onwards, $\chi_t = \chi_*$ in each $t \in [0, \infty)$.

[3] The intuition is that choosing a different consumption-capital ratio at time zero, $\chi_0 \gtrless \chi_*$, would generate – from equation (91) – explosive dynamics in χ_t which would violate either the consumers' transversality conditions in the long run (due to overaccumulation of capital) or the aggregate resource constraint (89) in finite time (due to overconsumption).

Using (11) and (74), the ratio between money-purchased and crypto-purchased goods (57) equals

$$\frac{c_t^m}{c_t^x} = \left[\frac{\theta\,(1+2\xi)}{1-\theta}\right]^\sigma. \tag{93}$$

Equation (92) allows us to the determine the ratio between money-purchased consumption and capital. Since aggregate consumption equals

$$C_t = c_t^m + c_t^x = c_t^m \cdot \left\{1 + \left[\frac{1-\theta}{\theta\,(1+2\xi)}\right]^\sigma\right\}, \tag{94}$$

the ratio $\chi_t^m \equiv c_t^m/K_t$ will be constant over time and equal to

$$\chi_t^m = \frac{c_t^m}{K_t} = \frac{1}{1+\left[\frac{1-\theta}{\theta(1+2\xi)}\right]^\sigma} \cdot \frac{C_t}{K_t} = \frac{1}{1+\left[\frac{1-\theta}{\theta(1+2\xi)}\right]^\sigma} \cdot \chi_*,$$

that is,

$$\chi^m = \frac{A - g^C}{1+\left[\frac{1-\theta}{\theta(1+2\xi)}\right]^\sigma}. \tag{95}$$

Expression (95) determines the variable χ^m that we have previously introduced in equation (68) and confirms that it is constant over time. Similarly, letting $\sigma = 1$, expression (95) determines the variable χ_{CD}^m that we have previously introduced in equation (47). We have now all the elements to build a reduced system for the BGP equilibrium in the AK model.

Reduced system (BGP equilibrium). The following *reduced equilibrium system* allows us to determine four key endogenous variables – namely, the inflation rate, the balanced growth rate (of real consumption, output and capital), the consumption-capital ratio, and employment in the final sector (and, residually, in the exchange platform) – along the balanced growth path of the AK model:

$$g^M = \frac{\sqrt{(1+\pi)^2 + 4\alpha A} - (1-\pi)}{2} - \rho \tag{96}$$

$$g^C = g^M - \pi \tag{97}$$

$$\chi^m = \frac{A - g^C}{1+\left[\frac{1-\theta}{\theta(1+2\xi)}\right]^\sigma} \tag{98}$$

$$\frac{L^y}{1-L^y} = \frac{A\,(1-\alpha)}{2\xi} \cdot \frac{1}{\chi^m} \cdot \left[\frac{\theta\,(1+2\xi)}{1-\theta}\right]^\sigma \tag{99}$$

Equation (96) follows immediately from (49) and determines the inflation rate π given the monetary growth rate g^M set by the authority. Equation (97) follows immediately from (48) and determines the BGP growth rate g^C. Equation (98) follows from the above analysis – eq.(95) – and determines the ratio of consumption in money-purchased goods to physical capital. Equation (99) follows from substituting (92) and (95) into the condition for wage equalization in the labor market (88), and determines employment in the final sector, L^y, as well as employment in the exchange platform, $1 - L^y$.

Since the economy exhibits a BGP equilibrium from time zero onwards, the determination of $\left(\pi, g^C, \chi^m, L^y\right)$ in the reduced system allows us to calculate the whole time paths of the main variables of interest according to the following equations: capital, output and consumption are given by

$$
\begin{aligned}
K_t &= K_0 \cdot e^{g_C \cdot t}, && (100)\\
Y_t &= AK_t = K_0 \cdot e^{g_C \cdot t}, && (101)\\
C_t &= (C_t/K_t) \cdot K_t = \chi_* \cdot K_t = \left(A - g^C\right) \cdot K_0 \cdot e^{g_C \cdot t}, && (102)
\end{aligned}
$$

whereas the real wage and price level are given by[4]

$$
\begin{aligned}
\frac{w_t}{P_t} &= (1-\alpha) \cdot A \cdot (1/L^y) \cdot K_0 \cdot e^{g_C \cdot t} && (103)\\
P_t &= \frac{1}{g^C + \chi^m \left[1 + \left(\frac{1-\theta}{\theta}\right)^{\sigma} \cdot (1+2\xi)^{1-\sigma}\right]} \cdot \frac{M_0}{K_0} \cdot e^{\pi t} && (104)
\end{aligned}
$$

where K_0 is exogenously given and M_0 is exogenously set by the authority.

8.5 BGP equilbrium: numerical analysis

The following subsection presents a numerical illustration of the balanced growth path for different values of the elasticity of substitution – covering the cases of strict complementarity, strict substitutability, and Cobb-Douglas preferences – and evaluates, for each of these three basline scenarios, the effects of exogenous changes in the growth rate of nominal money, in the crypto-fee rate (due to an exogenous rise in ξ), and in the taste parameter θ. First, we report the fixed parameter values and the results for the baseline scenrio in Table 6 and Table 7.

Table 6: Parameter values.

Preferences	**Technology**	**Monetary policy rule**
$\theta = 0.3$	$\alpha = 0.3$	$g^M = 0.045$
$\rho = 0.02$	$\xi = 0.05$	
	$A = 0.16$	

8.6 BGP shock analysis

An increase in g^M (faster monetary growth). A 10% increase in the money supply (g^M), as reported in Table 8, leads to a rise in inflation (π) across all cases, regardless of the elasticity of substitution

[4]The time path of the real wage in (103) follows straightforwardly from equation (41). The time path of the money price in (104) follows from equation (70) after substituting the equilibrium fee rate (74).

Table 7: Benchmark results.

	π	g^C	χ^m	L^y	$\frac{c_t^m}{c_t^x}$
$\sigma = 0.5$	0.0199	0.0251	0.0549	0.9333	0.6866
$\sigma = 1$	0.0199	0.0251	0.0432	0.9243	0.4714
$\sigma = 1.5$	0.0199	0.0251	0.0330	0.9166	0.3237

(σ). As expected, the real consumption growth rate (g^C) decreases, indicating the negative effect of higher inflation on real consumption. The ratio of money-purchased goods to capital (χ^m) increases, suggesting a shift towards more money-purchased goods as inflation rises. However, this increase is more pronounced when money and crypto are complements ($\sigma = 0.5$) and less so when they are substitutes ($\sigma = 1.5$). Labor allocation to the goods production sector (L^y) declines slightly as money becomes more abundant, reflecting a reallocation of labor resources. Lastly, the ratio of money-purchased to crypto-purchased goods (c_t^m/c_t^x) falls as σ increases, implying that when money and crypto-currency are substitutes, consumers favor crypto-purchased goods more heavily after the shock.

Table 8: Shock Analysis: 10% Increase in the Money Supply

	π	g^C	χ^m	L^y	$\frac{c_t^m}{c_t^x}$
$\sigma = 0.5$	0.0246↑	0.0249↓	0.0550↑	0.9332↓	0.6866
$\sigma = 1$	0.0246↑	0.0249↓	0.0433↑	0.9242↓	0.4714
$\sigma = 1.5$	0.0246↑	0.0249↓	0.0330↑	0.9165↓	0.3237

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 7. The absence of an arrow signifies no change compared to the benchmark.

An increase in ξ (which raises fees, δ). In Table 9, a 10% increase in the fee structure (ξ) leads to no change in inflation (π) and the consumption growth rate (g^C) across all cases, regardless of the elasticity of substitution (σ). However, the ratio of money-purchased goods to capital (χ^m) increases, indicating a shift towards money-purchased goods as the cost of crypto-related transactions rises. This increase in χ^m is larger when money and crypto are complements ($\sigma = 0.5$) and less so when they are substitutes ($\sigma = 1.5$). Labor allocation to the goods production sector (L^y) declines, reflecting a reduction in the productive sector as crypto becomes more costly to use. The ratio of money-purchased to crypto-purchased goods (c_t^m/c_t^x) rises, suggesting that higher fees for crypto transactions push consumers to favor money-purchased goods, with this effect being strongest when the two goods are more substitutable. This analysis highlights the role of transaction costs in shifting consumer preferences between money and crypto, and its impact on real variables in the BGP framework.

Table 9: Shock Analysis: 10% increase in the fee structure

	π	g^C	χ^m	L^y	$\frac{c_t^m}{c_t^x}$
$\sigma = 0.5$	0.0199	0.0251	0.0551↑	0.9273↓	0.6897↑
$\sigma = 1$	0.0199	0.0251	0.0435↑	0.9176↓	0.4757↑
$\sigma = 1.5$	0.0199	0.0251	0.0333↑	0.9093↓	0.3281↑

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 7. The absence of an arrow signifies no change compared to the benchmark.

A reduction in θ (higher taste for crypto-purchased goods). As reported in Table 10, a 10% reduction in θ results in no change in inflation (π) and consumption growth (g^C) across all values of the elasticity of substitution (σ). However, the ratio of money-purchased goods to capital (χ^m) decreases, indicating a shift toward crypto-purchased goods. This reduction in χ^m is more pronounced when the two goods are complements ($\sigma = 0.5$) and less significant when they are substitutes ($\sigma = 1.5$). Labor allocation to the goods production sector (L^y) also decreases, reflecting a reduced need for money-purchased goods as the economy adapts to the higher preference for crypto-currency transactions. The ratio of money-purchased to crypto-purchased goods (c_t^m/c_t^x) decreases sharply, showing that consumers are opting more for crypto-purchased goods, with the largest decline occurring when the goods are more substitutable ($\sigma = 1.5$). This shift highlights the influence of consumer preferences on the allocation of resources in the economy.

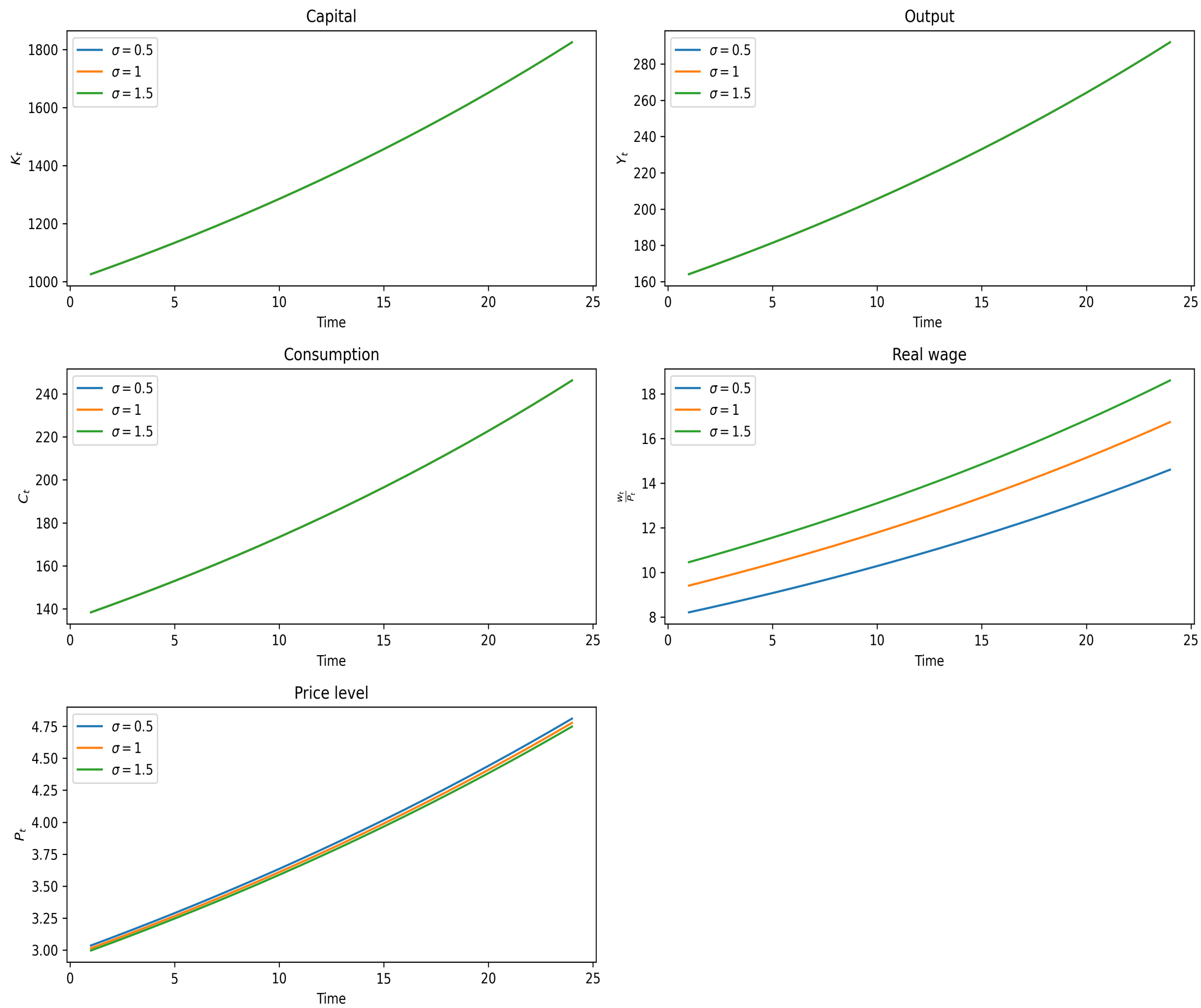
Table 10: Shock analysis: 10% decrease in θ

	π	g^C	χ^m	L^y	$\frac{c_t^m}{c_t^x}$
$\sigma = 0.5$	0.0199	0.0251	0.0525↓	0.9315↓	0.6378↓
$\sigma = 1$	0.0199	0.0251	0.0390↓	0.9211↓	0.4068↓
$\sigma = 1.5$	0.0199	0.0251	0.0278↓	0.9127↓	0.2595↓

Note: The upward (downward) arrow indicates an increase (decrease) relative to the benchmark values reported in Table 7. The absence of an arrow signifies no change compared to the benchmark.

Figure 1 demonstrates that capital, output, and consumption grow steadily over time but remain largely unaffected by variations in the elasticity of substitution ($\sigma = 0.5$, $\sigma = 1$, and $\sigma = 1.5$). This suggests that the broader growth trajectory of the economy is driven by technology rather than consumer preferences between money-purchased and crypto-purchased goods. However, real wages are highly sensitive to changes in σ, with greater substitutability ($\sigma = 1.5$) leading to faster wage growth due to more efficient labor allocation. In contrast, when goods are more complementary ($\sigma = 0.5$), wage growth is slower. The price level shows only slight variation, rising more slowly with greater substitutability, reflecting the lower cost pressures from crypto-purchased goods. Overall, the impact of elasticity is most visible in real wages, while price levels and aggregate economic variables remain relatively stable.

Figure 1: Evolution of the key model variables along the BGP



9 Conclusion

This paper explores how the coexistence of fiat money and crypto-currencies shapes economic outcomes in a dynamic setting. We highlight that crypto-currencies disrupt resource allocation, particularly by diverting labor from traditional sectors and adding transaction costs, amplifying the non-neutrality of money. While key growth indicators like capital and output remain relatively stable, shifts in labor allocation and real wages are more responsive to changes in crypto fees and consumer preferences. An important improvement on this work would be to incorporate the idea of pecuniary and non-pecuniary features in the dynamic framework. Up until now, we have assumed that crypto goods are needed. Although the consumptio ratio is determined endogenously, future research could take a similar approach to the static model to determine a threshold good where consumers are indifferent between payment methods. Moreover, extending the model to include Central Bank Digital Currencies (CBDCs) would provide a valuable avenue for studying interactions among fiat money, crypto-currencies, and CBDCs, along with their influence on consumer preferences. This extension would greatly enhance the current analysis.

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